Translations of Gaussian Measures for the Infinite Dimensional Segal-Bargmann Representation

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Abstract

Estimates from analysis in finite dimensions which do not depend on the dimension will occasionally continue to hold in infinite dimensions with respect to suitable measures. A classical example of this would be the log-Sobolev inequality on path space. However, the fact that no Lebesgue (i.e. translation-invariant) measure exists in infinite dimensions can cause problems if one wants to have well-behaved convolutions of functions in infinite dimensions. Nevertheless, we provide a proof and exposition of the classical Cameron-Martin theorem which demonstrates that the transformation law for translations of Gaussian measures in finite dimensions continues to hold for Gaussian measures in infinite dimensions with respect to a dense subspace of translation directions. This provides a first step towards understanding infinite dimensional versions of the Segal-Bargmann representation of Fock space as a space of holomorphic functions.

Contents

1 Analysis in Finite vs. Infinite Dimensions

Infinite dimensional manifolds occur quite frequently in geometry and mathematical physics. In order to perform analysis on them, one might hope to have a nice class of measures on these infinite dimensional spaces generalizing useful measures from finite dimensions. Before discussing the problem of constructing such measures, let's do some examples.

Example 1.1. *Let G be a compact Lie group and consider the Banach manifolds $C_I^0([0,1],G)$ and $C_I^0(S^1,G)$ of continuous paths and loops in G respectively, both based at the identity $I \in G$. The homotopy*

$$\begin{aligned} H : C_I^0([0,1],G) \times [0,1] &\to C_I^0([0,1],G) \\ (f,s) &\mapsto (f_s(t) := f(st)) \end{aligned}$$

demonstrates that $C_I^0([0,1],G)$ is contractible. The map $C_I^0([0,1],G) \to G$ given by $f \mapsto f(1)$ is a fibration whose fiber at $I \in G$ is $C_I^0(S^1,G)$. The long exact sequence in homotopy then tells us that

$$\pi_2(G) \cong \pi_1(C_I^0(S^1,G)).$$

It is calculus on the infinite dimensional space $C_I^0(S^1,G)$ [7] which then allows one to prove that its fundamental group is trivial and hence $\pi_2(G) = 0$.

While the above example used analysis to study the topology of a Lie group through its loop space, there is a much more involved result relating the geometry of a Riemannian manifold to the geometry of its path space.

Theorem 1.2. *[8]*
Given a smooth complete Riemannian manifold (M^n, g) equipped with the measure $e^{-f}dV_g$, the two-sided inequality

$$-\delta g \leq \mathrm{Ric}_g + \mathrm{Hess}(f) \leq \delta g$$

holds if and only if for any $F \in L^2(C^0([0,\infty), M), dW_f)$ we have

$$\left|\nabla \int_{C^0([0,\infty),M)} F(x) W_f(dx)\right| \leq \int_{C^0([0,\infty),M)} \left(|\nabla_{PT_0} F|^2 + \int_0^\infty \frac{\delta}{2} e^{\delta t/2} |\nabla_{PT_t} F| dt \right) W_f(dx)$$

where $|\nabla_{PT_t} F|(x)$ is the supremum of the gradients of F in unit direction fields that vanish at $x(s)$ for $s < t$ and are parallel to the path x after time t. W_f is the Wiener measure.

Aside from path spaces, there are also infinite dimensional manifolds with applications to gauge field theory.

Example 1.3. *Let G be a compact Lie group and fix a faithful representation $G \subseteq \mathrm{SO}(k)$, as well as an infinite dimensional separable Hilbert space $\mathcal{H}$. We denote*

$$V_k^\perp(\mathcal{H}) := \{(x_1, \ldots, x_k) \in \mathcal{H}^k \ : \ x_1, \ldots, x_k \textit{ are orthonormal}\}.$$

One can make use of the shift operator on $\mathcal{H}$ with respect to some fixed Hilbert space basis to demonstrate that $V_k^\perp(\mathcal{H})$ is contractible. Furthermore, the natural action of G on $V_k^\perp(\mathcal{H})$ is free and the quotient map gives us an infinite dimensional (Hilbert) principal G-bundle:

$$V_k^\perp(\mathcal{H}) \to V_k^\perp(\mathcal{H})/G.$$

Since the total space is contractible it follows that $V_k^\perp(\mathcal{H})/G = BG$ is the classifying space of G. That is to say: homotopy classes of maps from a smooth manifold M into BG correspond to isomorphism classes of principal G-bundles over M.

While it is certainly interesting to consider analysis on the above spaces with respect to some natural measures, the main motivation for us will come from quantum field theory (see [5] for example). When quantizing a classical system given by a Hamiltonian

$$H = \frac{1}{2}|p|^2 + V(q) : T^*\mathbb{R}^n \to \mathbb{R}$$

we replace H by an unbounded (self adjoint if V is nice enough) operator

$$\hat{H} = -\Delta_x + V(x) \ \text{ on } \ L^2(\mathbb{R}^n).$$

If we Taylor expand V near a minimum q_0:

$$V(q) = V(q_0) + \frac{1}{2} q^T \,\mathrm{Hess}_{q_0}(V) q + O(q^3)$$

then our Hamiltonian $\hat{H}$ can be approximated by a Harmonic oscillator Hamiltonian:

$$\hat{H}_0 = -\Delta_x + \frac{1}{2} x^T \,\mathrm{Hess}_{q_0}(V) x + V(q_0).$$

Diagonalizing the symmetric positive definite matrix $\mathrm{Hess}_{q_0}(V)$ so that its eigenvalues are $0 < \lambda_1^2/2 \leq \cdot \leq \lambda_n^2/2$ we can perform an orthogonal change of coordinates to coordinates y where

$$\hat{H}_0 = -\Delta_y + \frac{1}{4} \sum_{i=1}^n \lambda_i^2 x_i^2 + V(q_0).$$

Renormalizing by replacing $V(q_0)$ with $-\frac{1}{2}\operatorname{Tr}((2\operatorname{Hess}_{q_0}(V))^{1/2}) = -\frac{1}{2}(\lambda_1 + \cdots + \lambda_n)$ we arrive at

$$\hat{H}_0^{re} := \sum_{i=1}^{n} \left(-\partial_i^2 + \frac{1}{4}\lambda_i^2 x_i^2 - \frac{1}{2}\lambda_i \right)$$

This can be diagonalized by introducing the following "annihilation" and "creation" operators:

$$a_j := \lambda^{-1/2}\partial_j + \frac{1}{2}\lambda^{1/2}x_j$$
$$a_j^* := -\lambda^{-1/2}\partial_j + \frac{1}{2}\lambda^{1/2}x_j.$$

If we set

$$e_0(y) := (\lambda_1 \cdots \lambda_n/(2\pi)^n)^{1/4} \exp\left(-(\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2)/4\right)$$

then we see that as $\vec{k} \in \mathbb{N}^n$ ranges over all multi-indices we get a complete set of eigenfunctions:

$$e_{\vec{k}} := a_{k_1}^* \cdots a_{k_n}^* e_0 \ \text{ for } \hat{H}_0^{re} \text{ with eigenvalue } \vec{k}\cdot\vec{\lambda} = k_1\lambda_1 + \cdots + k_n\lambda_n.$$

Furthermore our Hamiltonian becomes

$$\hat{H}_0^{re} = \sum_{i=1}^{n} \lambda_i a_i^* a_i.$$

The point of our renormalization is that this operator is non-negative and actually has a kernel. This way, if we were to take $n \to \infty$ then the spectrum won't get "pushed out" to infinity. Similarly, this description of $\hat{H}_0^{re}$ allows us to replace $L^2(\mathbb{R}^n)$ with the Fock space $\mathcal{F}(\mathbb{C}^n)$ where a_j and a_j^* act as quantum analogues of complex coordinates. In fact, $\mathcal{F}(\mathbb{C}^n)$ can be viewed as the space of entire functions on $\mathbb{C}^n$ that are in L^2 with respect to a Gaussian measure.

In quantum field theory, say for a scalar field, one instead has a Hamiltonian of the form

$$H(\pi, \phi) = \frac{1}{2}\|\pi\|_{L^2}^2 + \int \left(\frac{1}{2}|\nabla\phi|^2 + W(\phi) \right)$$

and so, working perturbatively near a local minimum, our $\operatorname{Hess}(V)$ from the finite dimensional case is replaced by the operator $-\Delta$. We can make sense of the quantum version of the quadratic part of this Hamiltonian on Fock space $\mathcal{F}(\mathcal{H})$ where $\mathcal{H} = L^2$ here. One thing that could be asked is the following:

In this infinite dimensional context, can $\mathcal{F}(\mathcal{H})$ be realized as a space of holomorphic functions on $\mathcal{H}$ that are L^2 with respect to a Gaussian measure on $\mathcal{H}$?

It should be noted that the covariance matrix of this Gaussian measure (in both the finite dimensional and infinite dimensional contexts) will depend on the quadratic part of the Hamiltonian. If our scalar field ϕ lived on a Riemannian manifold (M^n, g) then this quadratic part is given by the Laplacian $-\Delta_g$ of g, which contains geometric information concerning (M, g) (such as the dimension and the volume of (M, g) via the Weyl law and trace formula). We could then ask what an infinite dimensional version of the below result would say.

Example 1.4. *Let (M^n, g) be a Riemannian manifold with heat kernel:*

$$\rho_t(x, y) = (4\pi t)^{-n/2} \exp\left(-\delta_{x,t}(y)/2\right)$$

and consider

$$\hat{H} := -\Delta_g + \frac{1}{4}|\nabla\delta_{x,t}|^2 - \frac{1}{2}\Delta_g \delta_{x,t}.$$

If $\operatorname{Ric}_g \geq 0$ then the log-Sobolev inequality implies this Hamiltonian has a discrete spectrum with spectral gap $1/t$. Furthermore, there is a partial converse to this result.

Given the technicalities involved with working in infinite dimensions we will stick to a very concrete example of a Gaussian measure on Hilbert spaces: the Wiener measure from probability.

The goal of this write-up is to investigate the behaviour of the Wiener measure under translations. In particular, given a continuous function u in the Wiener space $\mathcal{W}$ of continuous functions $u : [0, \infty) \to \mathbb{R}$ such that $u(0) = 0$, we can define a function $T_u : \mathcal{W} \to \mathcal{W}$ via $T_u(v) = u + v$. Then, if W is the Wiener meaure, we can ask whether it is possible to express the push-forward WT_u^{-1} of W along T_u in terms of W.

Before we begin, I have added an appendix on Gaussian cylinder set measures since the proof sketches of proposition 2.8 and theorem 3.1 are more believable when one understands the construction of the Wiener measure from a Gaussian cylinder set measure on the Cameron-Martin Hilbert space.

Now, one may recall that on $\mathbb{R}^n$ the n-dimensional Lebesgue measure is translation-invariant. That is to say that for any $y \in \mathbb{R}^n$, $\lambda^n T_y^{-1} = \lambda^n$ (from here on T_x will always denote translation by x so long as we are in a vector space and λ^n will denote the Lebesgue measure on $\mathbb{R}^n$). However, we can show that this is not the case for the Wiener measure W.

Definition 1.5. *A* **strictly positive measure** *on a topological space X is a Borel measure μ on X for which $\mu(U) > 0$ for all non-empty open subsets U of X.*

Proposition 1.6. *Let X be an infinite dimensional Banach space. If μ is a translation invariant strictly positive measure on X, then $\mu(U) = \infty$ for all non-empty open subsets $U \subseteq X$.*

Proof. For this proof, we will rely on the fact that in an infinite dimensional Banach space there exists a countably infinite collection $\{e_n\}_{n=1}^{\infty}$ of unit vectors with $\|e_n - e_m\| > 1/2$ whenever $m \neq n$. This fact is usually proven as a lemma to the proof that the closed unit ball in an infinite dimensional Banach space is not compact and I will omit the proof here in the interest of time. Now, suppose for contradiction that such a measure μ on X existed. Since μ is translation invariant it suffices to prove that $\mu(B_r(0)) = \infty$ for every $r > 0$. However, since rescaling the e_n's by $r/2$ still yields a uniform separation amongst them by $r/4$, it will actually suffice to show that $\mu(B_2(0)) = \infty$.

Now, if $n \neq m$ then $B_{1/2}(e_n) \cap B_{1/2}(e_m) = \emptyset$. Also, since μ is a positive translation invariant measure on X it follows that there exists some $c > 0$ (possibly infinite) so that $\mu(B_{1/2}(e_n)) = c$ for all n. But then, by σ-additivity:

$$\mu(B_2(0)) \geq \sum_{n=1}^{\infty} \mu(B_{1/2}(e_n)) = \infty.$$

□

Notice, however, that the above result does not immediately apply to $\mathcal{W}$. First of all, the Wiener space $\mathcal{W}$ is not a Banach space. Equipped with the seminorms:

$$\|u\|_{n,\infty} = \sup_{t \in [0,n]} |u(t)|, \quad \text{and the metric} \quad d(u,v) = \sum_{n=1}^{\infty} \frac{\max\{1, \|u - v\|_{n,\infty}\}}{2^n}$$

$\mathcal{W}$ becomes a Fréchet space and has the topology of uniform convergence on compact sets. Notice that if we define $\mathcal{W}_1 := \{u \in C^0([0,1], \mathbb{R}) : u(0) = 0\}$ then equipped with the supremum norm $\|\cdot\|_\infty$, $\mathcal{W}_1$ becomes a Banach space. Furthermore, we can consider the linear map:

$$\begin{aligned} \pi : \mathcal{W} &\to \mathcal{W}_1 \\ u &\mapsto u|_{[0,1]}. \end{aligned}$$

Given $\epsilon > 0$ we notice that if $u, v \in \mathcal{W}$ with $d(u,v) < \epsilon/2$ then $\|\pi(u) - \pi(v)\|_\infty = \|u - v\|_{1,\infty} \leq 2d(u,v) < \epsilon$ so $\pi : \mathcal{W} \to \mathcal{W}_1$ is indeed continuous. So, given the Wiener measure W on $\mathcal{W}$, we can define a measure $W_1 := W\pi^{-1}$ on $\mathcal{W}_1$ which we will call the Wiener measure on $\mathcal{W}_1$. This is now a measure on a Banach space and will be a useful tool in the next proposition.

Proposition 1.7. *The Wiener measure W on $(\mathcal{W}, \mathcal{B}_\mathcal{W})$ is not translation invariant.*

Proof. Suppose for contradiction that it was and consider the measure W_1 on the Banach space $(\mathcal{W}_1, \|\cdot\|_\infty)$. This is a strictly positive probability measure on $\mathcal{W}_1$ since W is a strictly positive probability measure on $\mathcal{W}$. Also, given $A \in \mathcal{B}_{\mathcal{W}_1}$ and $u \in \mathcal{W}_1$ we can define $v : [0,\infty) \to \mathbb{R}$ via $v(t) = u(t)$ for $t \in [0,1]$ and $v(t) = u(1)$ for $t > 1$. Notice that $\pi(v) = u$. Then we can compute:

$$\begin{aligned} W_1(A - u) &= W(\pi^{-1}(A - u)) = W(\{w \in \mathcal{W} \ : \ \pi(w) = a - \pi(v) \ \text{ for some } a \in A\}) \\ &= W(\{w \in \mathcal{W} : \pi(w + v) \in A\}) = W(T_v^{-1}(\pi^{-1}(A))) \\ &= W(\pi^{-1}(A)) = W_1(A). \end{aligned}$$

Therefore W_1 is a strictly positive translation invariant probability measure on a Banach space, a contradiction. $\square$

One remark is that, while perhaps unfortunate, the above result should not be discouraging. We had no reason to expect the Wiener measure to behave similarly to the Lebesgue measure since the Wiener measure was, in fact, modelled after Gaussian measures. Recall then that the standard Gaussian measure γ^n on $\mathbb{R}^n$ is given by:

$$\gamma^n(A) = \int_A \exp\left(-\frac{1}{2}\|x\|^2\right) d\lambda^n(x)$$

for all Borel $A \subseteq \mathbb{R}^n$. Now, given any $A \in \mathcal{B}_{\mathbb{R}^n}$ (the Borel sigma algebra) and any $y \in \mathbb{R}^n$ we can compute via a change of coordinates:

$$\begin{aligned} \gamma^n(A - y) &= \int_A \exp\left(-\frac{1}{2}\|x - y\|^2\right) d\lambda^n(x) = \int_A \exp\left(-\frac{1}{2}\langle x - y, x - y\rangle\right) d\lambda^n(x) \\ &= \int_A \exp\left(\langle x, y\rangle - \frac{1}{2}\|y\|^2\right) \exp\left(-\frac{1}{2}\|x\|^2\right) d\lambda^n(x) \end{aligned}$$

and taking the Radon-Nikodym derivative we obtain:

$$\frac{d\gamma^n T_y^{-1}}{d\lambda^n}(x) = \exp\left(\langle x, y\rangle - \frac{1}{2}\|y\|^2\right) \frac{d\gamma^n}{d\lambda^n}(x).$$

One remark worth making is that the Radon-Nikodym derivative is only defined almost everywhere so the above is actually a representative for the Radon-Nikodym derivative. Now, we cannot expect to generalize the above result to the case of the Wiener measure since there is no analogue to λ^n on $\mathcal{W}$. However, via the chain rule we can rewrite the above expression to obtain:

$$\frac{d\gamma^n T_y^{-1}}{d\gamma^n}(x) = \exp\left(\langle x, y\rangle - \frac{1}{2}\|y\|^2\right).$$

Also, the appearance of a norm and inner product in the above expression suggests that our generalization will be to some sort of Banach space or Hilbert space as opposed to the Fréchet space $\mathcal{W}$. This is why we introduced the space $(\mathcal{W}_1, \|\cdot\|_\infty)$ earlier, as it will actually be the setting for our generalization: the Cameron-Martin theorem.

2 A Functional-Analytic Description of the Wiener Integral

The computations in the previous section demonstrated that Gaussian measures on $\mathbb{R}^n$ behave quite well under translations despite not being translation invariant. One consequence of the expression we derived is that Gaussian measures have the same null sets as their translations. This property turns out to be fairly restrictive in the infinite dimensional case.

Definition 2.1. *A measure μ on a Banach space X is called* **pseudo translation invariant** *if and only if $\mu \ll \mu T_y^{-1} \ll \mu$ for all $y \in X$.*

Theorem 2.2. *There are no non-trivial pseudo translation invariant locally finite Borel measures on a separable infinite dimensional Banach space.*

In particular, since $\mathcal{W}_1$ is separable we cannot hope to obtain a nice formula for the Radon-Nikodym derivative $\frac{dW_1 T_u^{-1}}{dW_1}$ for every $u \in \mathcal{W}_1$. As such, we will restrict ourselves to looking at translations by elements in a sort of embedded Hilbert space in $\mathcal{W}_1$. This Hilbert space will in fact be a Sobolev space of sorts and so, in order to be self-contained, I will state several definitions and facts regarding the Sobolev space we desire.

Definition 2.3. *Let $f \in L^1([0,1],\mathbb{R})$. A* **weak derivative** *of f is an element $g \in L^1([0,1],\mathbb{R})$ so that for any $h \in C^\infty([0,1],\mathbb{R})$ with $h(0) = h(1) = 0$ we have:*

$$\int_0^1 f h' d\lambda = -\int_0^1 g h d\lambda.$$

Proposition 2.4. *Let $f \in L^1([0,1],\mathbb{R})$ and suppose that f has weak derivatives $g, h \in L^1([0,1],\mathbb{R})$. Then:*

1. *$g = h$ a.e. in $[0,1]$;*
2. *there exists an absolutely continuous function $u : [0,1] \to \mathbb{R}$ equal to f a.e. satisfying:*

$$g(x) = \lim_{t\to 0} \frac{u(x+t) - u(x)}{t} \quad \textit{for almost every } x \in [0,1].$$

In particular, since two continuous functions agree almost everywhere if and only if they agree everywhere, it follows that given any weakly differentiable $f \in L^1([0,1],\mathbb{R})$ there exists a unique absolutely continuous representative for f. As such, we identify f with its absolutely continuous representative and recall that absolutely continuous functions do indeed admit a derivative almost everywhere and this derivative is in L^1. Notice that this allows us to talk about weakly differentiable functions pointwise. Furthermore, we will denote the representative for the weak derivative of f arising from the absolutely continuous representative for f by f'.

Definition 2.5. *We denote by $W^{1,2}([0,1],\mathbb{R})$ the space of all weakly differentiable $f \in L^2([0,1],\mathbb{R})$ so that $f' \in L^2([0,1],\mathbb{R})$. We then denote by $\mathcal{H}$ the collection of all $f \in W^{1,2}([0,1],\mathbb{R})$ such that $f(0) = 0$. One can show that this $\mathcal{H}$ is a Hilbert space when equipped with the inner product:*

$$\langle f, g \rangle_{\mathcal{H}} = \int_0^1 f' g' d\lambda.$$

Now, seeing as the elements of $\mathcal{H}$ are really absolutely continuous functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$, we can consider the inclusion $\phi : \mathcal{H} \to \mathcal{W}_1$. By definition ϕ is linear and so we know ask whether it is continuous.

Proposition 2.6. *The inclusion $\phi : \mathcal{H} \to \mathcal{W}_1$ is continuous.*

Proof. Since we are working with a linear map between Banach spaces it suffices to prove boundedness. Now, given $f \in \mathcal{H}$ we notice that f is a continuous function on the compact set $[0,1]$ and so there exists a $t_0 \in [0,1]$ so that $\|f\|_\infty = |f(t_0)|$. But then we can compute:

$$\|f\|_\infty^2 = |f(t_0) - f(0)|^2 = \Big| \int_0^{t_0} f'(t)dt \Big|^2 \le \left(\int_0^1 |f'(t)|dt \right)^2 \le \int_0^1 |f'(t)|^2 dt = \langle f, f \rangle_{\mathcal{H}}$$

where the last inequality follows from Jensen's inequality. In particular, $\|\phi(f)\|_\infty \le \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and so $\|\phi\| \le 1 < \infty$, as required. $\square$

It turns out that $\phi(\mathcal{H}) \subseteq \mathcal{W}_1$ is the collection of functions along which we can understand translations of W_1. Now, if we think back to our result for the standard Gaussian measure on $\mathbb{R}^n$, we still have yet to find a substitute for "$\langle x, y \rangle$". It turns out that the correct approach is to replace this inner product with a Wiener integral of sorts. The Wiener integral is an isometry $L^2([0,\infty), \mathbb{R}) \to L^2(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, dW)$ however, as it turns out, the spaces $L^2([0,\infty), \mathbb{R})$ and $L^2([0,1], \mathbb{R})$ are too large for our purposes. Instead the Wiener integral which we will use is given by a map $\mathcal{H} \to L^2(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, dW_1)$.

Proposition 2.7. *Let $\phi^* : \mathcal{W}_1^* \to \mathcal{H}^*$ denote the Banach space adjoint of ϕ. Then ϕ^* is injective and $\phi^*(\mathcal{W}_1^*)$ is dense in $\mathcal{H}^*$.*

Proof. To prove this we recall some facts from functional analysis. A bounded operator is injective if and only if the range of its (Banach space) adjoint is dense in the codomain (I am omitting the proof of this since it requires the Hahn-Banach theorem). As such, $\overline{\phi^*(\mathcal{W}_1^*)} = \mathcal{H}^*$ since $\phi : \mathcal{H} \to \mathcal{W}_1$ is injective (it is an inclusion).

As for showing that ϕ^* is injective, recall that $\mathcal{H}$ is dense in $\mathcal{W}_1$ (which is a space of real-valued continuous functions on a compact interval equipped with the supremum norm) by the Stone-Weierstrass theorem. But then a bounded linear functional $g \in W_1^*$ is completely determined by its action on $\mathcal{H}$. Furthermore, ϕ^* is defined as $\phi^*(g)(h) = g(\phi(h)) = g(h)$ for all $h \in \mathcal{H}$ and so if $\phi^*(g) = \phi^*(f)$ for $f, g \in \mathcal{W}_1^*$ then g and f agree on $\mathcal{H}$ hence $g = f$. $\square$

Now, by the Riesz representation theorem for Hilbert spaces we have an isometric isomorphism $S : \mathcal{H}^* \to \mathcal{H}$ given by sending the functional $\langle \cdot, h \rangle$ to the element h. We can then consider the composition:

$$\psi := S \circ \phi^* : \mathcal{W}_1^* \to \mathcal{H}^* \to \mathcal{H}.$$

By our above proposition, ψ is an injective continuous linear map with dense image. This function will be instrumental in defining our new Wiener integral. Before doing this, however, we must prove a proposition in which we actually integrate over all of $\mathcal{W}_1$.

Proposition 2.8. *Suppose $f \in \mathcal{W}_1^*$. Then $f \in L^2_{\mathbb{R}}(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, dW_1)$. In fact, f is a centered Gaussian random variable with variance $\|\psi(f)\|_{\mathcal{H}}^2$.*

Proof. The proof here is quite technical so I will only provide a brief sketch. The idea is to approximate f by taking successive refinements of a division of $[0,1]$ and on each one defining a linear map which takes a continuous function $u \in \mathcal{W}_1$ and linearly interpolates u along our division. Then we obtain our approximations of f by pre-composing f with these linear interpolation operators. From here we use the fact that the Wiener measure can be constructed as a "radonification" of a Gaussian cylinder set measure on $\mathcal{H}$ along $\phi : \mathcal{H} \to \mathcal{W}_1$ to show that each of these approximations are centered Gaussian random variables and hence so is f.

In order to compute the variance, one applies the Riesz representation theorem for a space of continuous functions on a compact set to obtain an expression for f as a linear combination of integration functionals where the integrals are with respect to certain finite signed Radon measures. Then, one can write these measures as Lebesgue-Stieltjes measures and apply a generalized version of Itô's formula to obtain the desired variance for f. $\square$

Equipped with the above result we are now able to define our Wiener integral $\mathbb{W} : \mathcal{H} \to L^2(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, dW_1)$ by first declaring:

$$\mathbb{W}(\psi(f)) := [f] \in L^2(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, dW_1)$$

for all $f \in \mathcal{W}_1^*$. This is indeed well-defined by our last two propositions. As such, since $\psi(\mathcal{W}_1^*)$ is dense in $\mathcal{H}$ we can extend the above definition continuously to a map $\mathbb{W} : \mathcal{H} \to L^2_{\mathbb{R}}(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, W_1)$. At this point we are ready to both state and prove the Cameron-Martin theorem, however before doing so I hope to convince you that the above-defined map does indeed deserve the moniker of "Wiener integral". In order to do this, we recall that the Wiener integral $\mathrm{WI} : L^2([0,\infty), \mathbb{R}) \to L^2(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, dW)$ is often instead characterized by $\mathrm{WI}(\mathbf{1}_{[s,t)}) = X_t - X_s$ for all $s < t$ in $[0,\infty)$ where X_t and X_s are the evaluation maps at t and s respectively.

Now, suppose $s < t$ in $[0,1]$ and let $X_t, X_s : \mathcal{W}_1 \to \mathbb{R}$ be the evaluation maps at t, s respectively. Since $X_t - X_s \in \mathcal{W}_1^*$ we can ask ourselves: what element of $\mathcal{H}$ is sent to $X_t - X_s$ by $\mathbb{W}$? The way $\mathbb{W}$ is defined, it suffices to compute $\psi(X_t - X_s) = S(\phi^*(X_t - X_s))$. Now, $\phi^*(X_t - X_s)$ is the linear functional on $\mathcal{H}$ given by $\phi^*(X_t - X_s)(u) = (X_t - X_s)(\phi(u)) = u(t) - u(s)$. So now we need to find an element $h \in \mathcal{H}$ so that for all $u \in \mathcal{H}$ we have:

$$u(t) - u(s) = \langle u, h \rangle_{\mathcal{H}} = \int_0^1 u'(x) h'(x) dx.$$

Notice however that:

$$\int_0^1 u'(x) \mathbf{1}_{[s,t)}(x) dx = \int_s^t u'(x) dx = u(t) - u(s)$$

for all $u \in \mathcal{H}$. So, unravelling the definition of the weak derivative, we realize that we need to find a function $h \in \mathcal{H}$ so that:

$$\int_0^1 h(x)\varphi'(x) dx = -\int_0^1 \mathbf{1}_{[s,t)}(x)\varphi(x) dx = -\int_s^t \varphi(x) dx \text{ for all } \varphi \in C^\infty([0,1], \mathbb{R}) \text{ with } \varphi(0) = \varphi(1) = 0.$$

Now, consider $h(x) := -\int_x^1 \mathbf{1}_{[s,t)}(y) dy$. We do indeed have $h \in \mathcal{H}$ since h' exists and is in L^2 by the fundamental theorem of calculus. Also, given any $\varphi \in C^\infty([0,1], \mathbb{R})$ with $\varphi(0) = \varphi(1) = 0$ we can compute:

$$\begin{aligned}
\int_0^1 h(x)\varphi'(x) dx &= -\int_0^1 \int_x^1 \mathbf{1}_{[s,t)}(y)\varphi'(x) dy dx \\
&= -\int_0^1 \mathbf{1}_{[s,t)}(y) \int_0^1 \mathbf{1}_{[x,1)}(y)\varphi'(x) dx dy \text{ by Fubini's theorem} \\
&= -\int_s^t \varphi(y) dy.
\end{aligned}$$

So it follows that $\mathbb{W}$ sends $-\int_x^1 \mathbf{1}_{[s,t)}(y) dy \in \mathcal{H}$ to $X_t - X_s \in L^2(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1}, dW_1)$. But now, recall that for every $f \in \mathcal{H}$ we have $f' \in L^2([0,1], \mathbb{R})$. Furthermore $h' = \mathbf{1}_{[s,t)}$ and so $\mathbb{W}(h) = X_t - X_s = \mathrm{WI}(h')$. In fact, if we define:

$$\begin{aligned}
R : L^2([0,1], \mathbb{R}) &\to L^1([0,1], \mathbb{R}) \\
Rf(x) &= -\int_x^1 f(y) dy
\end{aligned}$$

then, by performing a similar computation to the one above, one can see that $Rf \in W^{1,2}([0,1], \mathbb{R})$ for all $f \in L^2([0,1], \mathbb{R})$ and $(Rf)' = f$. Therefore, by the characterization of the Wiener integral mentioned above, we actually have:

$$\mathrm{WI} = \mathbb{W} \circ R.$$

3 The Cameron-Martin Theorem

In this section, I will state and prove the Cameron-Martin theorem. The original proof due to Cameron and Martin is fairly technical, however it is also instructive and demonstrates the thought process behind how this formula was originally derived. To be precise, the proof was done by actually computing the Radon-Nikodym derivative of $W_1 T_h^{-1}$ with respect to W_1 (where $h \in \mathcal{H}$) via an approximation. The proof I will instead present here is newer and quite shorter, but relies on us knowing the correct formula beforehand.

Theorem 3.1. *Assume that we are within he framework of the previous section. For each $h \in \mathcal{H}$, consider the translation map $T_h : \mathcal{W}_1 \to \mathcal{W}_1$ given by $T_h(u) = u + \phi(h)$. Then $W_1 \ll W_1 T_h^{-1} \ll W_1$ and:*

$$\frac{dW_1 T_h^{-1}}{dW_1}(u) = \exp\left(\mathbb{W}(h)(u) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right).$$

i.e., for all $A \in \mathcal{B}_{\mathcal{W}_1}$ we have:

$$W_1 T_h^{-1}(A) = \int_A \exp\left(\mathbb{W}(h)(u) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right) dW_1(u).$$

Proof. This is merely a proof sketch (albeit a nearly complete one) since in order to present the full proof much more theory would have been needed first. Now, arbitrarily select $h \in \mathcal{H}$. Recall that two probability measures on the Wiener space $\mathcal{W}$ agree if and only if all of their finite dimensional marginals agree. We begin by defining a probability measure μ on $(\mathcal{W}_1, \mathcal{B}_{\mathcal{W}_1})$ via:

$$\mu(A) = \int_A \exp\left(\mathbb{W}(h)(u) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right) dW_1(u)$$

for all $A \in \mathcal{B}_{\mathcal{W}_1}$. Indeed, the reason for this is that one can construct the measure W_1 abstractly as the radonification of the canonical Gaussian cylinder set measure on $\mathcal{H}$ through the map $\phi : \mathcal{H} \to \mathcal{W}_1$, and then prove that W_1 is in fact equal to $W\pi^{-1}$; and the property of being characterized by finite dimensional marginals is encapsulated in the definition of a cylinder set measure. But then the finite dimensional marginals of both μ and $W_1 T_h^{-1}$ are probability measures on $\mathbb{R}^k$ (where k depends on the number of sampled points in the marginal). Now, the uniqueness of the Fourier transform, which tells us that two probability measures on $\mathbb{R}^k$ agree if and only if their fourier transforms agree. As such, it suffices to prove that for all $f \in \mathcal{W}_1^*$ we have:

$$\int_{\mathcal{W}_1} e^{if(u)} dW_1 T_h^{-1}(u) = \int_{\mathcal{W}_1} e^{if(u)} d\mu(u).$$

Through a change of variables we can compute:

$$\int_{\mathcal{W}_1} e^{if(u)} dW_1 T_h^{-1}(u) = \int_{\mathcal{W}_1} e^{if(u+\phi(h))} dW_1(u) = e^{if(\phi(h))} E[e^{if}] = e^{if(\phi(h))} e^{-\|\psi(f)\|_{\mathcal{H}}^2/2}$$

using what we know about the finite dimensional marginals, and that f is a centered Gaussian random variable with variance $\|\psi(f)\|_{\mathcal{H}}^2$. But now we can also use the fact that $\mu \ll W_1$ (since μ is defined by integrating with respect to W_1) as well as the Radon-Nikodym theorem to compute:

$$\begin{aligned}\int_{\mathcal{W}_1} e^{if(u)} d\mu(u) &= \int_{\mathcal{W}_1} \exp\left(if(u) + \mathbb{W}(h)(u) - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right) dW_1(u) \\ &= e^{-\|h\|_{\mathcal{H}}^2/2} \int_{\mathcal{W}_1} e^{if(u) + \mathbb{W}(h)(u)} dW_1(u).\end{aligned}$$

Now, consider the random vector $F : \mathcal{W}_1 \to \mathbb{R}^2$ given by $F(u) = (f(u), \mathbb{W}(h)(u))$. If h was actually equal to $\psi(g)$ for some $g \in \mathcal{W}_1^*$ then F would be a centered Gaussian random vector and we'd be able to use its covariance matrix in our computation. This is not the case in general, however $\overline{\psi(\mathcal{W}_1^*)} = \mathcal{H}$ so we can apply the Lebesgue dominated convergence theorem to obtain:

$$\int_{\mathcal{W}_1} e^{if(u)+\mathbb{W}(h)(u)} dW_1(u) = \int_{\mathcal{W}_1} e^{i\langle(1,-i),F(u)\rangle} dW_1(u) = e^{-\frac{1}{2}\|\psi(f)\|_{\mathcal{H}}^2 + if(\phi(h)) + \frac{1}{2}\|h\|_{\mathcal{H}}^2}$$

by the same techniques as before together with the computation of the characteristic functions of Gaussian random vectors in my previous notes, as well as the computation of the following covariance matrix:

$$\begin{pmatrix} \|f\|_{L^2}^2 & \langle f, \mathbb{W}(h)\rangle_{L^2} \\ \langle f, \mathbb{W}(h)\rangle_{L^2} & \|\mathbb{W}(h)\|_{L^2}^2 \end{pmatrix} = \begin{pmatrix} \|\psi(f)\|_{\mathcal{H}}^2 & f(\phi(h)) \\ f(\phi(h)) & \|h\|_{\mathcal{H}}^2 \end{pmatrix}.$$

So, in conclusion we have:

$$\int_{\mathcal{W}_1} e^{if(u)} dW_1 T_h^{-1}(u) = \exp\left(if(\phi(h)) - \frac{1}{2}\|\psi(f)\|_{\mathcal{H}}^2\right) = \int_{\mathcal{W}_1} e^{if(u)} d\mu(u) \quad \text{as required.}$$

$\square$

A Gaussian Cylinder Set Measures

In this section I will outline the construction of the Wiener measure W_1 on $\mathcal{W}_1$ via what is called the canonical cylinder set measure on a Hilbert space. This construction will hopefully make some of the above proof sketches more believable and will hopefully demonstrate the true importance of the Hilbert space $\mathcal{H}$, which is also known as the Cameron-Martin Hilbert space.

The idea here comes from defining the Wiener measure using its finite dimensional marginals. Given any pair (T, V_T) where V_T is a finite dimensional real vector space and $T : \mathcal{H} \to V_T$ is a continuous linear surjection, we can apply the first isomorphism theorem for Banach spaces to obtain:

$$V_T \cong \mathcal{H}/\ker T$$

and so V_T becomes a quotient inner product space $(V_T, \langle\cdot,\cdot\rangle_T$ via the inner product on $\mathcal{H}$. Now, if $n = \dim V_T$ and $S : \mathbb{R}^n \to V_T$ is the Hilbert space isomorphism taking $\mathbb{R}^n$ with the dot product to $(V_T, \langle\cdot,\cdot\rangle_T)$, then we can define the probability measure:

$$\gamma_T := \gamma^n S^{-1} \in \mathrm{Prob}(V_T)$$

where γ^n is the standard Gaussian on $\mathbb{R}^n$. Seeing that the finite dimensional marginals all arise from the continuous linear evaluation maps $X_{t_1,\dots,t_k} : \mathcal{W} \to \mathbb{R}^k$, it follows that the γ_T's should make a suitable replacement for the finite dimensional marginals in the context of the Cameron-Martin space $\mathcal{H}$. This motivates the following definition.

Definition A.1. *The* **canonical Gaussian cylinder set measure** *on $\mathcal{H}$ is the collection of all measures $\{\gamma_T \in \mathrm{Prob}(V_T) : (T, V_T) \in X\}$ where X is the collection of all pairs (T, V_T) where V_T is a finite dimensional real vector space and $T : \mathcal{H} \to V_T$ is a continuous linear surjection.*

A first thing to notice is that the canonical Gaussian cylinder set measure on $\mathcal{H}$ is not actually a measure on $\mathcal{H}$. One can ask however if there is some measure μ on $\mathcal{H}$ so that for all $(T, V_T) \in X$ we have $\mu T^{-1} = \gamma_T$. It turns out that this is not the case and this provides additional motivation for the usage of both $\mathcal{H}$ and $\mathcal{W}_1$ in the development of the Wiener integral defined previously.

Proposition A.2. *Let $C = \{T^{-1}(B) : B \in \mathcal{B}_{V_T},\ (T, V_T) \in X\}$. Then:*

1. *the Borel σ-algebra of $\mathcal{H}$ is equal to σ-$\mathrm{Alg}(C)$;*
2. *there is no measure μ on $\mathcal{H}$ so that $\mu T^{-1} = \gamma_T \in \mathrm{Prob}(V_T)$ for all $(T, V_T) \in X$.*

I am ommiting the proofs in this appendix, however they can be found in the references. Recall, though, that the Wiener measure W_1 is defined on $\mathcal{W}_1$ and not on $\mathcal{H}$. In fact, one can prove (similar to the proof of generic non-differentiability of Wiener paths) that $W_1(\phi(\mathcal{H})) = 0$. As such, the next theorem will explain precisely what I previously meant by $\phi : \mathcal{H} \to \mathcal{W}_1$ "radonifying" the Gaussian cylinder set measure.

Theorem A.3. *Let Y denote the collection of pairs (S, W_S) where each W_S is a finite dimensional real vector space and $S : \mathcal{W}_1 \to W_S$ is a continuous linear surjection. Then there exists a Borel probability measure W_0 on $\mathcal{W}_1$ so that for all $(S, W_S) \in Y$ we have:*

$$W_0 S^{-1} = \gamma_{S\circ\phi} \in \mathrm{Prob}(W_S).$$

The measure W_0 whose existence is asserted by the above theorem then turns out to actually be the Wiener measure W_1 on $\mathcal{W}_1$.

Theorem A.4. $W_0 = W_1 \in \mathrm{Prob}(\mathcal{W}_1)$.

Proof. Again this is merely a proof sketch. Let $0 < t_1 < t_2 < \cdots < t_k$ be in $[0, \infty)$ and consider the real random vector $X_{t_1,\ldots,t_k} : \mathcal{W} \to \mathbb{R}^k$ given by $X_{t_1,\ldots,t_k}(u) = (u(t_1), \ldots, u(t_k))$. For simplicity we write $T = \{t_1, \ldots, t_k\}$ and $X_T = X_{t_1,\ldots,t_k}$. Let $S = [0, 1] \cap T$. Since $W_1 = W\pi^{-1}$ it suffices to show that:

$$W_0 X_S^{-1} = W\pi^{-1} X_S^{-1}.$$

However, for any $A \in \mathcal{B}_{\mathcal{W}_1}$ we have $W\pi^{-1}X_S^{-1}(A) = W(X_S \circ \pi)^{-1}(A) = WX_S^{-1}(A)$ by the definition of π. However, W was defined so that WX_S^{-1} was a centered Gaussian random variable on $\mathbb{R}^{|S|}$ with a covariance matrix whose (i, j)'th entry was the minimum of t_i and t_j. But then the covariance matrix of $W_0 X_S^{-1} = \gamma_{X_S\circ\phi} = \gamma_{X_S} \in \mathrm{Prob}(\mathbb{R}^{|S|})$ can be computed explicitly and is equal to the covariance matrix of WX_S^{-1}. Then, by the last proposition in my previous writeup we indeed have $W_0 X_S^{-1} = WX_S^{-1}$ as required. $\square$

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